Representability Theorem.

We are finally ready to state a result that characterizes n-geometric stacks.

Thm (Lurie): Given a prestack $\mathcal{X}$, $\mathcal{X}$ is an n-geometric stack locally almost of finite type if and only if the following conditions are satisfied:

(i) $\mathcal{X}$ is laft (as a prestack);

(ii) $\mathcal{X}$ satisfies etale descent;

(iii) for any discrete complete local Noetherian k-algebra R_0 w/ maximal ideal $\mathfrak{m}_0 \subseteq R_0$ one has:

$$^{cl}\mathcal{X}(\operatorname{Spec} R_0) \xrightarrow{\simeq} \lim_{n\geq 1} {}^{cl}\mathcal{X}(\operatorname{Spec}(R_0/\mathfrak{m}_0^n)).$$

(iv) $\mathcal{X}$ admits deformation theory and $\forall x: S \to \mathcal{X}$ $T^*_x\mathcal{X} \in QCoh(S)^{\leq n}$, i.e. $T^*_x\mathcal{X}$ is $(-n)$-connective. (really in $Pro(QCoh(S)^{\leq n})$.)

(v) $^{cl}\mathcal{X}$ is n-truncated.

We start w/ some remarks:

Rk 1: The theorem as we stated it is close to its formulation in Lurie's theory (in S.C.R. set up). Though other versions were proved by him latter, SAG still does not have a proof of the above in the spectral setting.

Rk 2: The classical version of the above theorem (due to Artin) considers $\mathcal{X}_0$ a functor from $^{cl}Sch^{aff}$ to $Spc^{\leq 1}$ satisfying the natural analogues of (i-iii). Then (iv) corresponds to the conditions: $\mathcal{X}_0$ has an obstruction thy & def. thy and satisfies Schlessinger's criteria for formal representability (see Stacks-project for what this means.) The point we make though is that these are extra data than $\mathcal{X}_0$, they are not conditions on $\mathcal{X}_0$. (v) is not need, but one requires that $\mathcal{X}_0 \to \mathcal{X}_0 \times \mathcal{X}_0$ is representable by an algebraic space. For $\mathcal{X}$ the representability of the diagonal will be automatic from the other conditions.

Rk 3: One has a notion of complete local Noetherian derived ring $R \in CAlg$, i.e. there exists an unique $\mathfrak{m}_0 \subseteq H^0(R)$ and R is a Noeth. der. ring & $H^0(R) \xrightarrow{\simeq} \lim_{n\geq 1} H^0(R)/\mathfrak{m}_0^n$.

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One observes that (iii) is equivalent to:

(iii)$^{der.}$ For every local complete Noetherian $R \in CAlg$ one has:

$$\mathcal{X}(Spec\ R) \to \lim_{n \geq 1} \mathcal{X}(Spec(R_n)), \text{ where.}$$

$$R_n := R \otimes_{\mathbb{Z}[y_1,\dots,y_m]} \mathbb{Z} \quad \text{where. } x_1,\dots,x_m \text{ generate. } m_0 \underset{2^n}{\subseteq} H^0(R).$$
$$y_i \mapsto 0 \text{ in } \mathbb{Z} \ \& \ y_i \mapsto x_i \text{ in } H^0(R).$$

[DAG- Prop. 7.1.7.].

Moreover, since $\mathcal{X}$ admits deformation theory, we can relax condition (i) to.
(i)^{cl} ${}^{cl}\mathcal{X}$ is locally of finite type if we impose

(iv)$^{f.t.}$ $\mathcal{X}$ admits def. thy. & for every $x: S \to \mathcal{X}$ w/ $S \in {}^{cl}Sch^{aff}_{ft}$

$$T^*_x\mathcal{X} \in Pro(Coh(S)^-).$$

Let's first argue the necessity of conditions (i - v).

If $\mathcal{X}$ is an n-geometric stack laft then: (ii) follows from the def'n of a stack.
Condition (i) also follows from def'n. We didn't discuss this but a stack $\mathcal{X}$ is laft if $\forall n \geq 0$ ${}^{\leq n}\mathcal{X}$ is lft, i.e. ${}^{\leq n}\mathcal{X}$ is l.f.t. as a prestack. When ${}^{\leq n}\mathcal{X}$ is truncated, then is equivalent to require that ${}^{\leq n}\mathcal{X}$ is obtained by LKE of its restriction to ${}^{\leq n}Sch^{aff}_{ft}$. (technically of a stack in ${}^{\leq n}Sch^{aff}_{ft}$, so of the sheafification of its restriction.)

We argued conditions (iv) & (v) last time when we introduced n-geom. stacks. We are left w/.

<u>Claim:</u> For $\mathcal{X}$ an n-geometric stack, $\mathcal{X}$ satisfies (iii).

The proof consists in understanding $\mathcal{X}$ as a functor from the category of affine schemes étale over $S := Spec\ R$.

We introduce some notation. Let $s := \operatorname{Spec}(R/m_0) \to \operatorname{Spec} R = S$. and $s^{(n)} := \operatorname{Spec}(R/m_0^n)$. $n \geq 1$.

We let $\mathcal{X}|_{f.\acute{e}t.\,in\,S}$ denote the composite. $(\mathrm{Sch}^{aff}_{fin.\,\acute{e}t.\,in\,S})^{op} \to (\mathrm{Sch}^{aff})^{op} \xrightarrow{\mathcal{X}} \mathrm{Spc}$.
$\left(\begin{smallmatrix} T \to S \\ \text{finite étale} \end{smallmatrix}\right) \mapsto T \mapsto \mathcal{X}(T)$.

we let $\overline{\mathcal{X}} := (\mathcal{X}|_{f.\acute{e}t\,in\,S}) \times_S s$. denote $(\mathrm{Sch}^{aff}_{\acute{e}t.\,in\,s})^{op} \to (\mathrm{Sch}^{aff})^{op} \to \mathrm{Spc}$.
$t \to s \mapsto t \mapsto \mathcal{X}(t)$.

where we notice that any étale morphism ↑ is finite.

And since for each $n \geq 1$, one has: $\mathrm{Sch}^{aff}_{\acute{e}t.\,in\,s^{(n)}} \simeq \mathrm{Sch}^{aff}_{\acute{e}t.\,in\,s}$,

we let $\overline{\mathcal{X}}(n)$ be the composite: $(\mathrm{Sch}^{aff}_{\acute{e}t.\,in\,s})^{op} \simeq (\mathrm{Sch}_{\acute{e}t.\,in\,s^{(n)}})^{op} \to \mathrm{Spc}$.
$(t \to s) = (t^{(n)} \to s^{(n)}) \mapsto \mathcal{X}(t^{(n)})$.

We claim: $\mathcal{X}(S) \xrightarrow{\Theta} \mathcal{X}(s) = (\mathcal{X} \times_S s)(s)$ is an equivalence.

Indeed, one has this as consequence that R is Henselian, w/ maximal ideal m_0 & R/m_0 a field k.

$\{ \mathcal{Y} : (\mathrm{Sch}^{aff.}_{\acute{e}t.\,in\,S})^{op} \to \mathrm{Spc} \mid \mathcal{Y}$ is an étale sheaf & $\mathcal{Y} \xrightarrow{\simeq} \mathrm{RKE}_{(\mathrm{Sch}^{aff}_{f.\acute{e}t.\,in\,S} \hookrightarrow \mathrm{Sch}^{aff}_{\acute{e}t.\,in\,S})}(\mathcal{Y}|_{fin}). \}$

[SAG. B.6.5.3]. $\wr\!\wr$

$\{ \mathcal{Y}_0 : (\mathrm{Sch}_{\acute{e}t.\,in\,s})^{op} \mid \mathcal{Y}_0$ is an étale sheaf $\}$.

Then we claim. $\overline{\mathcal{X}}(t) \simeq \lim_{n \geq 1} \overline{\mathcal{X}}(n)(t)$ for any $t \to s$. étale.

Indeed, consider $\mathcal{Z} \to \mathcal{X}$ a cover. one has:

$\overline{\mathcal{X}}(t) = L(\overline{\mathcal{Z}}^{\bullet}/\overline{\mathcal{X}}|_{\acute{e}t\,t}(t)) \xrightarrow{\phi} L(\lim_{m \geq 1} \overline{\mathcal{Z}}^{\bullet}/\overline{\mathcal{X}}(m)) \xrightarrow{\phi'} \lim_{m \geq 1} \overline{\mathcal{X}}(m)$

the map ϕ' is an equivalence. iff $\lim_{m \geq 1} \overline{\mathcal{Z}}(m) \to \lim \overline{\mathcal{X}}(m)$ is an étale surjection.

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This follows from $\lim_{m\geq 1} \overline{\mathcal{Z}(m)} \to \lim_{m\geq 1} \overline{\mathcal{X}(m)}$ being a pullback of $\overline{\mathcal{Z}} \to \overline{\mathcal{X}}$ which is an étale surjection.

To check ϕ is an isomorphism we induce on n.
For $n>1$ this follows from the inductive hypothesis since $(\mathcal{Z}/\mathcal{X})^i$ is $(n-1)$-geometric $\forall i \geq 0$. So it is enough to consider $\underline{n=1}$.

Since each $(\mathcal{Z}/\mathcal{X})^i(R) \to (\mathcal{Z}/\mathcal{X})^0(R)$ for $i\geq 0$ is injective for R discrete we are reduced to checking it for $\mathcal{Z}(R)$ i.e. $\mathcal{Z} = \coprod_I U_i$ w/ U_i affine.

Finally, let $U_i = \mathrm{Spec}(A_i)$ and let $t = \mathrm{Spec}(B_0)$ and $R \to B$ étale finite s.t. $B_0 \simeq B \otimes_R R/m_0$. Then

$$\begin{array}{ccc}
\overline{\mathcal{X}}(t) & \longrightarrow & \lim_{m\geq 1} \overline{\mathcal{X}(m)}(t) \\
\| & & \wr\wr \\
\mathrm{Maps}(t, \mathrm{Spec}(R/m_0 \otimes_R A_i)) & & \lim_{m\geq 1} \mathrm{Maps}(\mathrm{Spec}(B \otimes_R R/m_0^i), U_i) \\
\wr\wr & & \wr\wr \\
\mathrm{Hom}^{\text{ét.}}_{\mathrm{CAlg}_{/R/m_0}}(R/m \otimes_R A_i, R/m \otimes_R B_0) & & \lim_{m\geq 1} \mathrm{Hom}_{\mathrm{CAlg}}(A_i, B \otimes_R R/m_0^i) \\
\wr\wr & & \wr\wr \\
\mathrm{Hom}_{\mathrm{CAlg}}(A_i, B) & & \mathrm{Hom}_{\mathrm{CAlg}}(A_i, \lim_{m\geq 1}(B \otimes_R R/m_0^i)) \\
 & & \wr\wr \\
\text{Since } R \to B \text{ is finite étale.} & & B.
\end{array}$$

$\blacksquare$

<u>RK</u>: The result we just sketched the proof is sometimes refered to Grothendieck's formal GAGA Theorem. The classical statement is that for any $m_0 \subseteq R$ as above:

$$\mathcal{X}_0(\mathrm{Spec}\, R) \to \mathcal{X}_0(\mathrm{Spf}\, R) \quad \text{is an equivalence.}$$

This follows by bootstrapping from the general fact that

$$\mathrm{Maps}(\mathrm{Spec}\, R, \mathrm{Spec}\, A) \xrightarrow{\sim} \mathrm{Maps}(\mathrm{Spf}\, R, \mathrm{Spec}\, A) \quad \text{for}$$

R a weakly admissible topological k-algebra. [See [Stacks-project, § 85.29].

We will try to sketch some of the ideas in the proof of this result. However, for a complete picture the reader should consult [SAG - Chapter 7].

First we flush at the notion of formal smoothness for n-geom morphisms.

Lemma: Let $f: \mathcal{X} \to \mathcal{Y}$ be an n-geometric morphism, then TFAE:

(i) f is smooth;

(ii) f is laft & $\forall S \xrightarrow{x} \mathcal{X}$ $T^*_x(\mathcal{X}/\mathcal{Y}) \in \mathrm{Perf}(S)^{\geq 0}$;

(iii) f is laft & $\forall S \xrightarrow{x} \mathcal{X}$ & $\mathcal{F} \in QCoh(S)^{\heartsuit}$ one has:

$$T^*_x(\mathcal{X}/\mathcal{Y}) \in \mathrm{Perf}(S) \ \& \ \mathrm{Hom}_{nQCoh(S)}(T^*_x(\mathcal{X}/\mathcal{Y}), \mathcal{F}[i]) = 0 \quad \forall i \geq 1.$$

(iv) f is laft & $\forall S \hookrightarrow S'$ a nilp. embedding. $\pi_0\mathcal{X}(S') \twoheadrightarrow \pi_0(S)$.

Idea: (i) $\Leftrightarrow$ (ii) by bootstrapping the result from before for affine schemes.
(ii) $\Leftrightarrow$ (iii) is a spectral seq. argument for $\mathrm{Ext}^i_{QCoh(S)}(T^*_x(\mathcal{X}/\mathcal{Y}), \mathcal{F})$ analogous to the relation between $M \in \mathrm{Perf}(R)_{[6,0]}$ & $M \in \mathrm{Mod}_R^{\leq 0}$. (See [Antieau-Gepner, Prop. 2.13].). ∎

The main tool in the proof is the following technical lemma. Essentially it says that to find a formally smooth map from an affine scheme to a laft prestack w/ def'thy. it is enough to control the $H^{-1}(-)$ of the relative cotangent complex and up to Zariski localization we can find a formally smooth map.

Tech Lemma: Suppose $\mathcal{X}$ satisfies (i) & (iv) and let $f_0: V_0 \to \mathcal{X}$ be a morphism from an affine scheme V_0 and $x: \mathrm{Spec}\, k \to V_0$ a point s.t. $H^{-1}(T^*_x(V_0/\mathcal{X})) = 0$.

Then there exists a Zariski nbd of x $\overline{V_0}$ s.t. $\overline{f_0}: \overline{V_0} \to \mathcal{X}$ factors as:

$$\begin{tikzcd} \overline{V_0} \arrow[r] \arrow[dr, "\overline{f_0}"'] & V \arrow[d, "f"] \\ & \mathcal{X} \end{tikzcd}$$

where (i) ${}^{cl}\overline{V_0} \simeq {}^{cl}V$;
(ii) f is formally smooth.

Moreover, if V_0 is almost of finite type (aft) then so is V.

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Additionally, if ${}^{cl}\mathcal{X}$ is 0-truncated then we can find $V \xrightarrow{f} \mathcal{X}$ formally étale, i.e. $\forall S \xrightarrow{y} V$ one has $T^*_y(V/\mathcal{X}) = 0$.

Idea: Use the flexibility of passing to ét. nbd. & the relation between vanishing of cotgt & inf. ext. to kill the cot. complex inductively.

Prop: Assume $\mathcal{X}$ satisfies (i-v) then there exists $U = \coprod_I T_i$ and

$$U \to \mathcal{X} \quad \text{a smooth surjection.}$$

Idea: Define $U := \coprod_I T_i$ where $f: T_i \to \mathcal{X}$ is a smooth map.
I all isom. classes of such f & T_i is aft.

So one only need to check $\varphi: U \to \mathcal{X}$ is surjective.

Let $x: S \to \mathcal{X}$ be a point in $\mathcal{X}$.
Since φ is formally smooth it is enough to consider $x: {}^{cl}S \to \mathcal{X}$.

B/c $\mathcal{X}$ is laft it is enough to take ${}^{cl}S$ of f.t. $/k$., i.e. ${}^{cl}S = \operatorname{Spec} A$
w/ A a disc. Noeth. k-algebra.
Since the statement is local for the étale topology we need to check that for a closed point $x: \operatorname{Spec} k \to {}^{cl}S$, there exists V_x an (étale) nbd. of x s.t.
$V_x \to T_i \to \mathcal{X}$ for some T_i as above.

Let V_x be the Henselianization at x in ${}^{cl}S$. Then $V_x \to \operatorname{Spec} k$ factors via $V_x \to W \to \operatorname{Spec} k$ where $W \simeq \mathbb{A}^n_k$.

Consider $\hat{W}_w$ the completion of W at w the image of x. Notice $\hat{W}_w$ is of f.t. $/k$.
The crucial point then is: Claim: One can modify $\hat{W}_w$ to Z_0 s.t.

- $\operatorname{Spec} k \to V_x \to Z_0 \to \mathcal{X}$, with $\operatorname{Spec} k \xrightarrow{z_0} Z_0$ - Z_0 is of f.t.
- ${}^{\leq 2}V_x \simeq {}^{\leq 2}Z_0$
- $H^{-1}(T^*_{z_0}(Z_0/\mathcal{X})) = 0$.

Then Tech. Lemma $\Rightarrow$ one can find $V_x \to Z_0 \to Z$ formally smooth

Let $\mathcal{X}$ satisfy (i-v).

Pf of Thm: Assume $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is m-geometric for some $m \in \mathbb{N}$. Then b/c a smooth surj. $p: U \to \mathcal{X}$ exists, we need to prove p is $(n-1)$-geometric. For any $S \to \mathcal{X}$ affine we obtain.

$S \times_{\mathcal{X}} U$ is $(m-1)$-geometric. Since ${}^{cl}\mathcal{X}$ is n-truncated the long. exact seq.

$$\cdots \to \pi_i(S \times_{\mathcal{X}} U({}^{cl}T)) \to \pi_i((S \times U)({}^{cl}T)) \to \pi_i({}^{cl}\mathcal{X}({}^{cl}T)) \to \pi_{i-1}(S \times_{\mathcal{X}} U({}^{cl}T)) \to \cdots$$

implies $m \leq n$.

Since any $(m-1)$-geometric morphism is $(n-1)$-geometric for $n \geq m$. This proves $\mathcal{X}$ is n-geometric.

Let's now prove that $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ being m-geometric for some m follows from the assumptions (i-v).

By induction on n it is enough to consider the case $n=0$.

If $\exists$ $\mathcal{Y}$ a 0-geometric stack s.t. $\pi_0 \mathcal{X}({}^{cl}S) \to \pi_0 \mathcal{Y}({}^{cl}S)$ then $\mathcal{X}$ is 0-geometric.

This follows from using Tech. Lemma in the ${}^{cl}\mathcal{X}$ 0-truncated case.

Finally, $\mathcal{Y}$ can be given by $U \times_{\mathcal{X}} U \to U \times U$. ∎

Rk: One can also parse the conditions (i-v) in their derived geometry part & their higher stacks part. I.e.

$\left(\mathcal{X}\text{ is laft } n\text{-geometric stack} \right) \Leftrightarrow \left(\mathcal{X} \text{ admits def'n theory } \& \ {}^{cl}\mathcal{X} \text{ is a classical } n\text{-geometric stack locally of finite type (lft).} \right)$

where the notion of a classical n-geometric stack is defined analogously to n-geometric stacks but starting w/ classical stacks & classical affine schemes.